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Some rules on resistance distance with applications

Yujun Yang and Heping Zhang[1]

School of Mathematics and Statistics, Lanzhou University, Lanzhou 730000, People's Republic of China

E-mail: yangyj03@lzu.cn and zhanghp@lzu.edu.cn

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Abstract
In this work, some rules for resistance distances of a graph G are established. Let S be a set of vertices of G such that all vertices in S have the same neighborhood N in $G - S$. If $|S| = 2, 3, 4$, simple formulae are derived to compute resistance distances between vertices in S in terms of the cardinality of N. These show that resistance distances between vertices in S depend only on the cardinality of N and the induced subgraph $G[S]$. One question arises naturally: does this property hold for S with arbitrarily many vertices? We answer this question by the following reduction principle: resistance distances between vertices in S can be computed as in the subgraph obtained from $G[S \cup N]$ by deleting all the edges between vertices in N.

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1. Introduction

On the basis of electrical network theory, a novel distance function, resistance distance as the effective resistance between two vertices, was identified by Klein and Randić [11] a decade back. Let G be a connected graph with vertex set V and edge set E. Suppose that vertices in G are labeled as $1, 2, \ldots, v$, where $v = |V|$. Then the resistance distance between vertices i and j, denoted by r_{ij} (if more than one graph is considered, we use r_{ij}^G to avoid confusion), is defined to be the effective resistance between them as computed with Ohm's law when all the edges of G are considered to be unit resistors.

As an important component of electrical circuit theory, effective resistance has been extensively studied in physics and engineering. Meanwhile, as an intrinsic graph metric and a relevant tool to characterize wave- or fluid-like communication between two vertices [9], it is well studied in mathematical and chemical literatures [1–4, 7, 10, 12, 14, 15, 19–24].

[1] Author to whom all the correspondence should be addressed.

From now on, we denote the (i, j)-entry of a matrix M by m_{ij}. Now we introduce some terminologies in graph theory and matrix theory. For a vertex i, let d_i denote the degree of i. If $U \subset V$ is any set of vertices, then $G[U]$ denotes the subgraph induced by U and $G - U$ denotes the graph obtained from G by deleting all the vertices in U and their incident edges. The adjacency matrix A of graph G is a $v \times v$ matrix with a_{ij} equal to 1 if vertices i and j are adjacent and 0 otherwise. Let $D = \mathrm{diag}\,(d_1, d_2, \ldots, d_v)$ be the diagonal matrix of vertex degrees. The Laplacian matrix of G is $L = D - A$. A generalized inverse of a matrix A, denoted by A^+, is a matrix such that $AA^+A = A$.

It is well known that L for a connected graph G has all positive eigenvalues except one that is 0. Hence L is singular and does not have an inverse. However, L does have a generalized inverse L^+. A fundamental formula to compute resistance distance is given in terms of the generalized inverse of the Laplacian matrix in [11]:

$$r_{ij} = l_{ii}^+ + l_{jj}^+ - l_{ij}^+ - l_{ji}^+. \tag{1}$$

Besides the generalized inverse of the Laplacian matrix, resistance distance can also be computed in terms of Laplacian eigenvalues and eigenvectors [8], normalized Laplacian eigenvalues and eigenvectors [5], spanning trees and spanning bi-trees [17] and random walks on graphs [6]. Conversely, some important parameters, such as expected hitting times in random walks on graphs, can be computed by an electrical approach using resistance distance [18].

In this paper, first of all, we obtain a pair of formulae (theorems 2.2 and 2.3) analogous to theorem C established in [11] (also theorem B in [9]), which may be viewed as general rules for resistance distance, with different particular realizations for different particular choices of M. In the following, by different choices of M, we establish a series of rules (theorems 3.1, 3.3 and 3.6). As an application, for any given set S with two or three or four vertices, if vertices in S have the same neighborhood N in $G - S$, then resistance distances between vertices in S can be easily computed in terms of the cardinality of N (theorems 3.2, 3.5 and 3.8). Motivated by these results, we consider S with arbitrarily many vertices and obtain the interesting reduction principle which can greatly simplify the calculation of resistance distances between vertices in S: if $S \subset V$ satisfies that all vertices in S have the same neighborhood N in $G - S$, then resistance distances between vertices in S can be computed as in the subgraph obtained from $G[S \cup N]$ by deleting all the edges between vertices in N.

2. General rules

For a square matrix M, let $\mathrm{Tr}(M)$ denote the trace of M, i.e. the sum of diagonal elements of M. Klein obtained the following result:

Theorem 2.1. *[9] For a v-vertex graph G and an arbitrary $v \times v$ matrix M,*

$$\sum_{i,j \in V} (LML)_{ij} r_{ij} = -2\,\mathrm{Tr}(ML). \tag{2}$$

In fact, theorem 2.1 was established earlier by Klein and Randić in [11], but in that article the matrix M was required to be symmetric.

Following the method used by Klein for proving theorem 2.1, we now obtain a pair of results analogous to theorem 2.1.

Theorem 2.2. *For a v-vertex graph G and an arbitrary $v \times v$ matrix M such that every column sums to* 0,

$$\sum_{i,j\in V} (ML)_{ij} r_{ij} = -2\,\mathrm{Tr}(M). \tag{3}$$

Proof. Substituting equation (1) into the left-hand side of equation (3) and bearing in mind that L^+ is symmetric (here L^+ is the Moore–Penrose inverse of L), we have

$$\begin{aligned}
&\sum_{i,j\in V} (ML)_{ij}\left(l^+_{ii} + l^+_{jj} - l^+_{ij} - l^+_{ji}\right) \\
&\quad = \sum_{i\in V} l^+_{ii} \sum_{j\in V} (ML)_{ij} + \sum_{j\in V} l^+_{jj} \sum_{i\in V} (ML)_{ij} - 2\sum_{i\in V}\sum_{j\in V} (ML)_{ij} l^+_{ji} \\
&\quad = \sum_{i\in V} l^+_{ii} M_i L 1 + \sum_{j\in V} l^+_{jj} 1^T M L^T_j - 2\sum_{i\in V} (MLL^+)_{ii}
\end{aligned}$$

where 1 is the vector with each element equal to 1, and M_i denotes the ith row of M. Since 1 is the 0-eigenvalue eigenvector to L and every column of M sums to 0, the first two terms vanish. On the other hand, by the well-known property

$$LL^+ = I - \frac{1}{v} J, \tag{4}$$

and the fact that $JM = \mathbf{0}$ since every column of M sums to 0, where J is the matrix with all elements equal to 1 and $\mathbf{0}$ is the zero matrix, we obtain that

$$-2\sum_{i\in V} (M(I-J))_{ii} = -2\,\mathrm{Tr}(M(I-J)) = -2\,\mathrm{Tr}((I-J)M) = -2\,\mathrm{Tr}(M).$$

Hence the proof is complete. □

In the same way, we can prove the following result:

Theorem 2.3. *For a v-vertex graph G and an arbitrary $v \times v$ matrix M such that every row sums to* 0,

$$\sum_{i,j\in V} (LM)_{ij} r_{ij} = -2\,\mathrm{Tr}(M). \tag{5}$$

Remark 1. The left-hand side of equation (2) in theorem 2.1 is symmetric and the right-hand side takes the trace of ML into account. The obtained pair of results in the present paper relate the trace of M respectively to the sum of all elements of ML and LM, weighted by the resistance distances.

Remark 2. Theorems 2.2 and 2.3 can also be derived from theorem 2.1 directly. State the relation in theorem 2.1 as

$$\sum_{i,j\in V} (LXL)_{ij} r_{ij} = -2\,\mathrm{Tr}(XL), \tag{6}$$

where X is an arbitrary $v \times v$ matrix. For any $v \times v$ matrix M each of whose columns sums to 0, put $X = L^+M$ in equation (6). We have

$$\sum_{i,j\in V} (LL^+ML)_{ij} r_{ij} = -2\,\mathrm{Tr}(L^+ML) = -2\,\mathrm{Tr}(LL^+M), \tag{7}$$

where the second equality holds since for any two $v \times v$ square matrices A and B,

$$\mathrm{Tr}(AB) = \sum_{i=1}^{v}\sum_{j=1}^{v} a_{ij}b_{ji} = \sum_{j=1}^{v}\sum_{i=1}^{v} b_{ji}a_{ij} = \mathrm{Tr}(BA).$$

Substituting equation (4) into equation (7), we arrive at

$$\sum_{i,j\in V} (ML)_{ij} r_{ij} = -2\,\mathrm{Tr}(M).$$

Hence theorem 2.2 is derived from theorem 2.1. Similarly, theorem 2.3 can also be derived from theorem 2.1 by putting $X = ML^{+}$ for M any matrix each of whose rows sums to 0.

3. Other rules and applications

For a vertex i, the neighborhood of i in a graph G, denoted by $N_G(i)$, is defined to be the set of all vertices adjacent to i in G. It should be pointed out that i may not belong to G.

On the basis of the above theorems, by particular choices of M, we can obtain a series of results in the following. For convenience, we divide the rest of this section into three parts.

3.1. The case of two vertices

In this part, we consider any two vertices i and j. Let $S = \{i, j\}$.

Theorem 3.1. *Let i, j be vertices of G. Then*

$$(d_i + d_j) r_{ij} + \sum_{k\in N_G(i)\setminus N_G(j)} (r_{ik} - r_{jk}) + \sum_{k\in N_G(j)\setminus N_G(i)} (r_{jk} - r_{ik}) = 4. \tag{8}$$

Proof. We choose M in theorem 2.2 such that $m_{ii} = m_{jj} = 1, m_{ij} = m_{ji} = -1$ and other elements are all 0. Then our result can be obtained directly. □

In particular, if i and j have the same neighborhood N in $G-S$, i.e. $N_{G-S}(i) = N_{G-S}(j) = N$, then it is straightforward to obtain the following result according to theorem 3.1.

Theorem 3.2. *Let i, j be vertices of G such that they have the same neighborhood N in $G - S$ and let $|N| = n$. If i and j are adjacent, then*

$$r_{ij} = \frac{2}{n+2}, \tag{9}$$

otherwise,

$$r_{ij} = \frac{2}{n}. \tag{10}$$

For example, we compute resistance distances for the complete graph K_v and complete bipartite graph $K_{u,v}$ by theorem 3.2. For K_v and $\forall\, i, j \in V$, by equation (9),

$$r_{ij} = \frac{2}{v-2+2} = \frac{2}{v}.$$

This coincides with the result obtained by Lukovits *et al* in [13]. For $K_{u,v}$, suppose that V_1 (resp. V_2) is the color set with u (resp. v) vertices. For $\forall\, i, j \in V_1$, by equation (10),

$$r_{ij} = \frac{2}{v}.$$

Similarly, we can obtain that for $\forall\, i, j \in V_2, r_{ij} = \frac{2}{u}$. This allows us to verify the results obtained by Klein in [9].

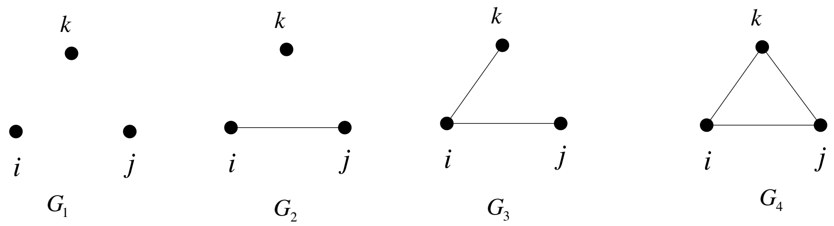

Figure 1. The isomorphically distinct graphs on three vertices.

3.2. The case of three vertices

Now we consider any three vertices i, j and k. Let $S = \{i, j, k\}$.

Theorem 3.3. *Let i, j, k be vertices of G. Then*

$$\sum_{s=1}^{v}((l_{js} - l_{ks})r_{is} + (l_{is} - l_{js})r_{js} + (l_{ks} - l_{is})r_{ks}) = 0. \tag{11}$$

Proof. The result follows from theorem 2.2 by choosing M such that $m_{ij} = m_{ji} = m_{kk} = 1$, $m_{ik} = m_{ki} = m_{jj} = -1$ and other places are all 0. □

In the following, we also consider the special case that i, j and k have the same neighborhood N in $G - S$, i.e. $N_{G-S}(i) = N_{G-S}(j) = N_{G-S}(k) = N$.

Theorem 3.4. *Let i, j, k be vertices of G such that they have the same neighborhood N in $G - S$. Then*

$$(d_i + d_j - l_{ij} - l_{jk})r_{ij} - (d_i + d_k - l_{ik} - l_{jk})r_{ik} + (l_{ik} - l_{ij})r_{jk} = 0. \tag{12}$$

Two graphs G and H are *isomorphic*, written $G \cong H$, if there exists a bijection $\theta : V(G) \to V(H)$ such that $ij \in E(G)$ if and only if $\theta(i)\theta(j) \in E(H)$ for all $i, j \in V(G)$.

As shown in figure 1, there are four isomorphically distinct graphs on three vertices. For $1 \leqslant i \leqslant 4$, if $G[S] \cong G_i$, we assume that the corresponding vertices under the isomorphism have the same labeling. In the following, r_{ij}, r_{ik} and r_{jk} are obtained in terms of the cardinality of N according to different cases of $G[S]$.

Theorem 3.5. *Let i, j, k be vertices of G such that they have the same neighborhood N in $G - S$ and let $|N| = n$.*

(i) If $G[S] \cong G_1$, then

$$r_{ij} = r_{ik} = r_{jk} = \frac{2}{n};$$

(ii) if $G[S] \cong G_2$, then

$$r_{ij} = \frac{2}{n+2}, \qquad r_{ik} = r_{jk} = \frac{2n+3}{n(n+2)};$$

(iii) if $G[S] \cong G_3$, then

$$r_{ij} = r_{ik} = \frac{2n+3}{(n+1)(n+3)}, \qquad r_{jk} = \frac{2}{n+1};$$

(iv) if $G[S] \cong G_4$*, then*

$$r_{ij} = r_{ik} = r_{jk} = \frac{2}{n+3}.$$

Proof. (i) and (iv) can be obtained directly by equations (9) and (10), respectively. Now we prove (ii). If $G[S] \cong G_2$, then it is obvious that $r_{ik} = r_{jk}$ by symmetry. Since i, j satisfy the condition of theorem 3.2, hence the first equality of (ii) holds. On the other hand, by equation (12) and the fact that $d_i = d_j = n+1, d_k = n$, we have

$$(2n+3)r_{ij} - (2n+1)r_{ik} + r_{jk} = 0. \tag{13}$$

Substituting $r_{ij} = \frac{2}{n+2}$ into equation (13) and solving for r_{ik}, we find the second equality of (ii). (iii) can be proved in a similar way as the proof of (ii) by theorems 3.2 and 3.4. □

3.3. The case of four vertices

Now we consider any four vertices i, j, k and l. Let $S = \{i, j, k, l\}$.

Theorem 3.6. *Let* i, j, k, l *be vertices of* G*. Then*

$$\sum_{s=1}^{v}((l_{ls} - l_{ks})r_{is} + (l_{ks} - l_{js})r_{js} + (l_{js} - l_{is})r_{ks} + (l_{is} - l_{ls})r_{ls}) = 4. \tag{14}$$

Proof. The proof proceeds from theorem 2.2 with the choice of M such that $m_{il} = m_{li} = m_{jk} = m_{kj} = 1, m_{ik} = m_{ki} = m_{jj} = m_{ll} = -1$ and other elements are all 0. □

In particular, if i, j, k and l have the same neighborhood N in $G - S$, i.e. $N_{G-S}(i) = N_{G-S}(j) = N_{G-S}(k) = N_{G-S}(l) = N$, we immediately have the following result according to theorem 3.6.

Theorem 3.7. *Let* i, j, k, l *be vertices of* G *such that they have the same neighborhood* N *in* $G - S$*. Then*

$$\begin{aligned}(l_{ik} + l_{jl} - l_{ij} - l_{jk})r_{ij} + (l_{ij} + l_{kl} - d_i - d_k)r_{ik} + (d_i + d_l - l_{il} - l_{kl})r_{il} \\ + (d_j + d_k - l_{ij} - l_{jk})r_{jk} + (l_{ij} + l_{kl} - 2l_{jl})r_{jl} + (l_{ik} + l_{jl} - l_{il} - l_{kl})r_{kl} = 4\end{aligned} \tag{15}$$

By graph-theoretical terminology, there are 11 isomorphically distinct graphs on four vertices as shown in figure 2. For $1 \leqslant i \leqslant 11$, if $G[S] \cong H_i$, we assume that the corresponding vertices under the isomorphism have the same labeling. In the following, resistance distances between pairs of vertices in S are obtained, which are expressed in terms of the cardinality of N.

Theorem 3.8. *Let* i, j, k, l *be vertices of* G *such that they have the same neighborhood* N *in* $G - S$ *and let* $|N| = n$*.*

(i) If $G[S'] \cong H_1$*, then*

$$r_{ij} = r_{ik} = r_{il} = r_{jk} = r_{jl} = r_{kl} = \frac{2}{n};$$

(ii) if $G[S'] \cong H_2$*, then*

$$r_{ij} = \frac{2}{n+2}, \qquad r_{kl} = \frac{2}{n}, \qquad r_{ik} = r_{il} = r_{jk} = r_{jl} = \frac{2n+3}{n(n+2)};$$

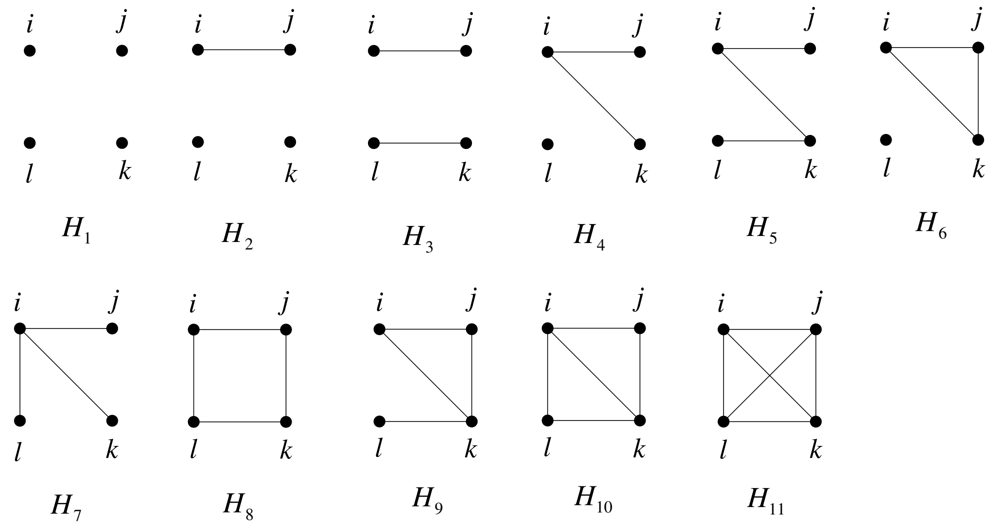

Figure 2. The isomorphically distinct graphs on four vertices.

(iii) if $G[S'] \cong H_3$*, then*

$$r_{ij} = r_{kl} = \frac{2}{n+2}, \qquad r_{ik} = r_{il} = r_{jk} = r_{jl} = \frac{2(n+1)}{n(n+2)};$$

(iv) if $G[S'] \cong H_4$*, then*

$$r_{ij} = r_{ik} = \frac{2n+3}{(n+1)(n+3)}, \qquad r_{jk} = \frac{2}{n+1}, \qquad r_{il} = \frac{2(n+2)}{n(n+3)},$$

$$r_{jl} = r_{kl} = \frac{2n^2+7n+4}{n(n+1)(n+3)};$$

(v) if $G[S'] \cong H_5$*, then*

$$r_{ij} = r_{kl} = \frac{2n^2+7n+4}{(n+2)(n^2+4n+2)}, \qquad r_{il} = r_{jk} = \frac{2n^2+9n+8}{(n+2)(n^2+4n+2)},$$

$$r_{ik} = \frac{2(n+1)}{n^2+4n+2}, \qquad r_{jl} = \frac{2(n+3)}{n^2+4n+2};$$

(vi) if $G[S'] \cong H_6$*, then*

$$r_{ij} = r_{ik} = r_{jk} = \frac{2}{n+3}, \qquad r_{il} = r_{jl} = r_{kl} = \frac{2(n+2)}{n(n+3)};$$

(vii) if $G[S'] \cong H_7$*, then*

$$r_{ij} = r_{ik} = r_{il} = \frac{2(n+2)}{(n+1)(n+4)}, \qquad r_{jk} = r_{jl} = r_{kl} = \frac{2}{n+1};$$

(viii) if $G[S'] \cong H_8$*, then*

$$r_{ij} = r_{jk} = r_{kl} = r_{il} = \frac{2(n+3)}{(n+2)(n+4)}, \qquad r_{ik} = r_{jk} = \frac{2}{n+2};$$

(ix) if $G[S'] \cong H_9$*, then*

$$r_{ij} = \frac{2}{n+3}, \qquad r_{kl} = \frac{2(n+2)}{(n+1)(n+4)},$$

$$r_{il} = r_{jl} = \frac{2n+5}{(n+1)(n+3)}, \qquad r_{ik} = r_{jk} = \frac{2n^2+9n+8}{(n+1)(n+3)(n+4)};$$

(x) if $G[S'] \cong H_{10}$*, then*

$$r_{ij} = r_{jk} = r_{kl} = r_{il} = \frac{2n+5}{(n+2)(n+4)}, \qquad r_{ik} = \frac{2}{n+4}, \qquad r_{jl} = \frac{2}{n+2};$$

(xi) if $G[S'] \cong H_{11}$*, then*

$$r_{ij} = r_{ik} = r_{il} = r_{jk} = r_{jl} = r_{kl} = \frac{2}{n+4}.$$

Proof. We only prove (v) and (ix), which are somewhat more complicated than the others to prove. The others can be proved in a similar way.

(v) If $G[S'] \cong H_5$, then by symmetry $r_{ij} = r_{kl}$ and $r_{il} = r_{jk}$. Suppose that $r_{ij} = r_{kl} = r_1, r_{il} = r_{jk} = r_2, r_{ik} = r_3$ and $r_{jl} = r_4$. Note that $d_i = d_k = n+2$ and $d_j = d_l = n+1$. By applying theorem 3.1 respectively to pairs of vertices $\{i, j\}, \{j, k\}, \{i, l\}$ and $\{j, l\}$, we can obtain the following four equations:

$$(2n+5)r_1 - r_2 + r_3 = 4, \tag{16}$$
$$r_1 - r_2 + (n+3)r_3 = 2, \tag{17}$$
$$r_1 + (2n+3)r_2 - r_4 = 4, \tag{18}$$
$$r_1 - r_2 - (n+1)r_4 = 2. \tag{19}$$

Solving the linear system formed by equations (16)–(19), we can obtain the desired result.

(ix) If $G[S'] \cong H_9$, then by symmetry $r_{ik} = r_{jk}$ and $r_{il} = r_{jl}$. Note that $d_i = d_j = n+2, d_l = n+1$ and $d_k = n+3$. By theorem 3.2, we readily have

$$r_{ij} = \frac{2}{n+3}.$$

Suppose that $r_{kl} = r_1, r_{ik} = r_{jk} = r_2$ and $r_{il} = r_{jl} = r_3$. By equation (15), we have

$$r_{ij} + (2n+2)r_3 = 4.$$

Hence,

$$r_3 = \frac{2n+5}{(n+1)(n+3)}.$$

By applying theorem 3.1 respectively to pairs of vertices $\{i, k\}$ and $\{k, l\}$, we have the following system of two equations:

$$r_1 + (2n+7)r_2 = 4 + r_3, \qquad (n+3)r_1 + r_2 = 2 + r_3.$$

Solving this system for r_1 and r_2, we have our desired result. □

For example, we compute resistance distances between vertices in S in the graph G as shown in figure 3. It is obvious that $G[S] \cong H_5$ and $n = 2$. Then by theorem 3.8 (v), it is easy to obtain that

$$r_{12} = r_{34} = \tfrac{13}{28}, \qquad r_{13} = r_{24} = \tfrac{17}{28}, \qquad r_{14} = \tfrac{5}{7}, \qquad r_{23} = \tfrac{3}{7}.$$

Theorems 2.1, 2.2 and 2.3 can be viewed as general rules. The results obtained in the above and in [9] are all derived from them by particular choices of M. We believe that some more interesting results may be obtained by other choices of M.

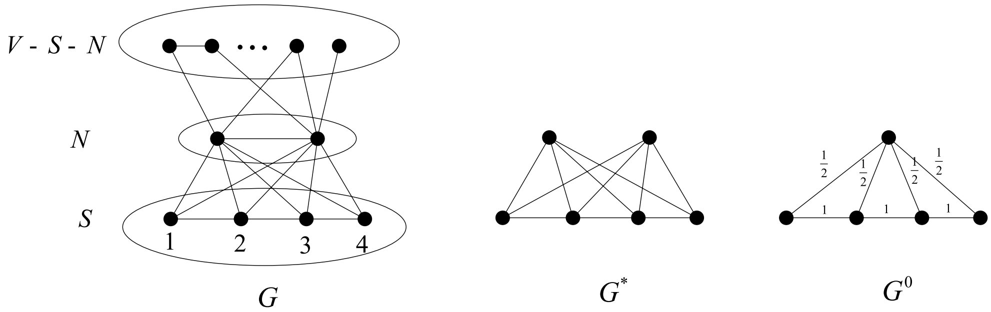

Figure 3. The graph G and the corresponding graphs G^* and G^0.

4. The reduction principle

Given a set S of two or three or four vertices, if vertices in S have the same neighborhood N in $G - S$, then by theorems 3.2, 3.5 and 3.8, resistance distances in S can be computed in terms of the cardinality of N, in other words, they can be uniquely determined by the cardinality of N and the subgraph $G[S]$. Does this proposition hold for S with arbitrarily many vertices? More precisely, if $S \subset V$ such that vertices in S have the same neighborhood N in $G - S$, can resistance distances between vertices in S be uniquely determined by the cardinality of N and the subgraph $G[S]$? In what follows, we give a positive answer to this question.

Before stating our results, we recall three important properties about effective resistance in electrical network theory, which will be used later.

(i) *Serial connection rule*: resistors that are connected in series can be replaced by a single resistor whose resistance is the sum of resistors.
(ii) *Parallel connection rule*: resistors that are connected in parallel can be replaced by a single resistor whose conductance (the inverse of resistance) is the sum of conductances.
(iii) *Rayleigh's monotonicity law* [16]: the effective resistance between any two vertices is a nondecreasing function of the edge resistances.

Let G^* be the graph obtained from $G[S \cup N]$ by deleting all the edges between vertices in N. For example, see figure 3. Then we have the following result.

Theorem 4.1. *Let S and G^* be defined as above. Then for $i, j \in S$,*

$$r_{ij}^{G} = r_{ij}^{G^*}.$$

Proof. We first show that our result holds if $G[V \backslash S]$ is complete. Impose a unit voltage between vertices i and j. Voltages $v(k)$ will be established at $k = 1, 2, \ldots, v$. We claim that vertices in N are at the same voltage. To see this, let k and l be any two vertices of N. By Ohm's law, the current i_{xy} that flows from any vertex x to its neighbor y is equal to

$$i_{xy} = \frac{v(x) - v(y)}{1} = v(x) - v(y).$$

By Kirchhoff's current law stating that the total current outflow any vertex is 0, we have

$$\sum_{\substack{x=1\\x\neq k}}^{v} i_{kx} = \sum_{\substack{x=1\\x\neq k}}^{v} (v(k) - v(x)) = 0.$$

Solving for $v(k)$ gives

$$v(k) = \frac{1}{v-1}\sum_{\substack{x=1\\x\neq k}}^{v} v(x).$$

In the same way, we can obtain that

$$v(l) = \frac{1}{v-1}\sum_{\substack{x=1\\x\neq l}}^{v} v(x).$$

So

$$v(k) - v(l) = \frac{1}{v-1}\sum_{\substack{x=1\\x\neq k}}^{v} v(x) - \frac{1}{v-1}\sum_{\substack{x=1\\x\neq l}}^{v} v(x) = \frac{1}{v-1}(v(l) - v(k)),$$

this gives $v(k) = v(l)$ and the claim is proved. Therefore, there is no current in the edges connecting the vertices in N and these edges can be omitted. In the same way as the proof of the claim, we can prove that vertices in $V\backslash(S \cup N)$ are at the same voltage and edges connecting them can be omitted too. Furthermore, we will show that there is no current in the edges between vertices in N and $V\backslash(S \cup N)$. Suppose that the voltages of vertices in N and $V\backslash(S \cup N)$ are equal to v_1 and v_2, respectively. If there are currents in these edges, this means that $v_1 \neq v_2$. If $v_1 > v_2$, currents flow from N to $V\backslash(S \cup N)$, but cannot flow back to N, which is impossible since all the currents should eventually flow to j. Otherwise, $v_2 > v_1$ and currents flow from $V\backslash(S \cup N)$ to N, which is also impossible since currents flow from i and they cannot reach $V\backslash(S \cup N)$ before N. Hence there is still no current in the edges connecting N and $V\backslash(S \cup N)$ and these edges can also be omitted. Meanwhile, vertices in $V\backslash(S \cup N)$ can be omitted since no current flows into or out of them. Hence, $r_{ij}^{G} = r_{ij}^{G^*}$.

Now suppose that $G[V\backslash S]$ is not complete. We construct a new graph G' from G by adding new edges between nonadjacent vertices of $V\backslash S$, i.e. $G'[V\backslash S]$ is complete. For $i, j \in S$, as proved above, $r_{ij}^{G'} = r_{ij}^{G^*}$. By Rayleigh's monotonicity law, on the one hand, $r_{ij}^{G} \geqslant r_{ij}^{G'}$ and on the other hand, $r_{ij}^{G} \leqslant r_{ij}^{G^*}$. Hence we have $r_{ij}^{G} = r_{ij}^{G^*}$ as well. □

By theorem 4.1, we can obtain the following interesting proposition, which may be viewed as a reduction principle.

Proposition 4.2 (the reduction principle). *If $S \subset V$ satisfies that all vertices in S have the same neighborhood N in $G - S$, then resistance distances between vertices in S can be computed as in the subgraph obtained from $G[S \cup N]$ by deleting all the edges between vertices in N.*

Remark 3. In fact, if we modify the resistances of some edges, then we can compute resistance distances between vertices in S in an even simpler graph than G^*. In this case, we should view a graph as a weighted graph such that each edge is assigned a weight. Resistance on each edge is equal to the weight assigned to this edge. Obviously graphs considered before can also be viewed as weighted graphs with weight 1 assigned to each edge. Since one can easily see that all vertices of N in G^* are at the same potential, we can identify them as a single vertex v_0. In the resulting graph, v_0 is connected to every vertex of S by n edges, where n is the cardinality of N. By the parallel connection rule, we can replace these edges by a single edge with resistance $\frac{1}{n}$. Let G^0 be the graph obtained from G^* by first contracting N to a single vertex v_0 (loops and multiple edges are deleted) and then changing the weights of edges connecting v_0 and vertices in S to $\frac{1}{n}$. For example, see figure 3. Then by theorem 4.1, we have

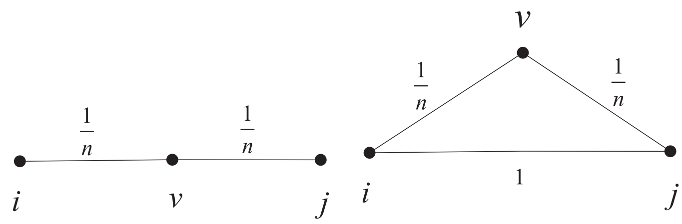

Figure 4. Illustration for G^0 corresponding to G in theorem 3.2.

Theorem 4.3. *Let S and G^0 be defined as above. Then for $i, j \in S$,*

$$r_{ij}^{G} = r_{ij}^{G^0}.$$

By theorem 4.3, we can obtain the following result, which may be viewed as another reduction principle.

Proposition 4.4. *If $S \subset V$ satisfies that all vertices in S have the same neighborhood N in $G - S$, then resistance distances between vertices in S can be computed as in the subgraph obtained from G by contracting $V - S$ to a single vertex v_0 (loops and multiple edges are deleted) and changing the resistances of edges connecting v_0 and vertices in S to $\frac{1}{n}$.*

Note that G^0 is a subgraph of G and can be handled in an easier way. An example is theorem 3.2, which can be viewed as a trivial consequence of proposition 4.4. Clearly, G^0 in the condition of theorem 3.2 is shown in figure 4, and the number on each edge is the weight (resistance) on it. The left one is the case that i and j are not adjacent and the right one is that they are adjacent. By serial and parallel connection rules, equations (9) and (10) can be easily obtained.

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